

TRACES ON FINITE \mathcal{W} -ALGEBRAS

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ABSTRACT. We compute the space of Poisson traces on a classical \mathcal{W} -algebra, i.e., linear functionals invariant under Hamiltonian derivations. Modulo any central character, this space identifies with the top cohomology of the corresponding Springer fiber. As a consequence, we deduce that the zeroth Hochschild homology of the corresponding quantum \mathcal{W} -algebra modulo a central character identifies with the top cohomology of the corresponding Springer fiber. This implies that the number of irreducible finite-dimensional representations of this algebra is bounded by the dimension of this top cohomology, which was established earlier by C. Dodd using reduction to positive characteristic. Finally, we prove that the entire cohomology of the Springer fiber identifies with the so-called Poisson-de Rham homology (defined previously by the authors) of the centrally reduced classical \mathcal{W} -algebra.

1. INTRODUCTION

The main goal of this note is to compute the zeroth Poisson homology of classical finite \mathcal{W} -algebras, and the zeroth Hochschild homology of their quantizations. Modulo any central character, both spaces turn out to be isomorphic to the top cohomology of the corresponding Springer fiber. The proof is based on the presentation of the Springer D -module on the nilpotent cone by generators and relations (due to Hotta and Kashiwara), and earlier results of the authors on the characterization of zeroth Poisson homology in terms of D -modules. This implies an upper bound on the number of irreducible finite-dimensional representations of a quantum \mathcal{W} -algebra with a fixed central character, which was previously established by C. Dodd using positive characteristic arguments. We also show that the Poisson-de Rham homology groups of a centrally reduced classical \mathcal{W} -algebra (defined earlier by the authors) are isomorphic to the cohomology groups of the Springer fiber in complementary dimension.

1.1. Definition of classical \mathcal{W} -algebras. We first recall the definition of classical \mathcal{W} -algebras (see, e.g., [GG02, Los10] and the references therein). Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} with the nondegenerate invariant form $\langle -, - \rangle$. We will identify \mathfrak{g} and \mathfrak{g}^* using this form. Let G be the adjoint group corresponding to \mathfrak{g} . Fix a nilpotent element $e \in \mathfrak{g}$. By the Jacobson-Morozov theorem, there exists an \mathfrak{sl}_2 -triple (e, h, f) , i.e., elements of \mathfrak{g} satisfying $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. For $i \in \mathbb{Z}$, let \mathfrak{g}_i denote the h -eigenspace of \mathfrak{g} of eigenvalue i . Equip \mathfrak{g} with the skew-symmetric form $\omega_e(x, y) := \langle e, [x, y] \rangle$. This restricts to a symplectic form on \mathfrak{g}_{-1} . Fix a Lagrangian $\mathfrak{l} \subset \mathfrak{g}_{-1}$, and set

$$(1.1) \quad \mathfrak{m}_e := \mathfrak{l} \oplus \bigoplus_{i \leq -2} \mathfrak{g}_i.$$

Then, we define a shift of \mathfrak{m}_e by e :

$$(1.2) \quad \mathfrak{m}'_e := \{x - \langle e, x \rangle : x \in \mathfrak{m}_e\} \subset \text{Sym } \mathfrak{g}.$$

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The classical \mathcal{W} -algebra \mathcal{W}_e is defined to be the Hamiltonian reduction of \mathfrak{g} with respect to \mathfrak{m}_e and the character $\langle e, \cdot \rangle$, i.e.,

$$(1.3) \quad \mathcal{W}_e := (\mathrm{Sym} \mathfrak{g} / \mathfrak{m}'_e \cdot \mathrm{Sym} \mathfrak{g})^{\mathfrak{m}_e},$$

where the invariants are taken with respect to the adjoint action. It is well known that, up to isomorphism, \mathcal{W}_e is independent of the choice of the \mathfrak{sl}_2 -triple containing e .

Since it is a Hamiltonian reduction, \mathcal{W}_e is naturally a Poisson algebra. The bracket $\{, \} : \mathcal{W}_e \otimes \mathcal{W}_e \rightarrow \mathcal{W}_e$ is induced by the standard bracket on $\mathrm{Sym} \mathfrak{g}$. The Poisson center of \mathcal{W}_e (i.e., elements z such that $\{z, F\} = 0$ for all F) is isomorphic to $(\mathrm{Sym} \mathfrak{g})^{\mathfrak{g}}$, by the embedding $(\mathrm{Sym} \mathfrak{g})^{\mathfrak{g}} \hookrightarrow (\mathrm{Sym} \mathfrak{g})^{\mathfrak{m}_e} \rightarrow (\mathrm{Sym} \mathfrak{g} / (\mathfrak{m}'_e \mathrm{Sym} \mathfrak{g}))^{\mathfrak{m}_e}$. It is known that this composition is injective (since, by Kostant's theorem, the coset $e + \mathfrak{m}_e$ meets generic semisimple coadjoint orbits of \mathfrak{g}).

Let $Z := (\mathrm{Sym} \mathfrak{g})^{\mathfrak{g}}$ and $Z_+ = (\mathfrak{g} \mathrm{Sym} \mathfrak{g})^{\mathfrak{g}}$ be its augmentation ideal. We therefore have an embedding $Z \hookrightarrow \mathcal{W}_e$, and can consider the central quotient

$$(1.4) \quad \mathcal{W}_e^0 := \mathcal{W}_e / Z_+ \mathcal{W}_e.$$

1.2. The Springer correspondence. We need to recall a version of the Springer correspondence between representations of the Weyl group W of \mathfrak{g} and certain G -equivariant local systems on nilpotent orbits in \mathfrak{g} .

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let \mathcal{B} be the flag variety of \mathfrak{g} , consisting of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. Consider the Grothendieck-Springer map $\rho : \tilde{\mathfrak{g}} := \{(\mathfrak{b}, g) : g \in \mathfrak{b}\} \subset \mathcal{B} \times \mathfrak{g} \rightarrow \mathfrak{g}$, which restricts to the Springer resolution $\tilde{\mathcal{N}} := \rho^{-1}(\mathcal{N}) \rightarrow \mathcal{N}$. Note that $\tilde{\mathcal{N}} \cong T^*\mathcal{B}$.

Let W be the Weyl group of \mathfrak{g} and $\mathrm{Irrep}(W)$ its set of irreducible representations, up to isomorphism. For each $\chi \in \mathrm{Irrep}(W)$, denote by V_χ the underlying vector space and by $\chi : W \rightarrow \mathrm{Aut}(V_\chi)$ the corresponding representation.

Then, there is a well known isomorphism (e.g., [Spr78, Theorem 1.13])

$$(1.5) \quad H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e)) \cong \bigoplus_{\chi \in \mathrm{Irrep}_e(W)} \psi_\chi \otimes V_\chi,$$

where $\mathrm{Irrep}(W) = \bigsqcup_{e \in \mathcal{N}/G} \mathrm{Irrep}_e(W)$, and for all $\chi \in \mathrm{Irrep}_e(W)$, ψ_χ is a certain irreducible representation of the component group $\pi_0(\mathrm{Stab}_G(e))$ of the stabilizer of e in G . For each $\chi \in \mathrm{Irrep}_e(W)$, let us use the notation $\mathbb{O}_\chi := \mathbb{O}(e) = G \cdot e$.

1.3. The main results. For any Poisson algebra A , we consider the *zeroth Poisson homology*, $\mathrm{HP}_0(A) := A / \{A, A\}$ (which is the same as the zeroth Lie homology). Its dual is the space of *Poisson traces*, i.e., linear functionals $A \rightarrow \mathbb{C}$ which are invariant under Hamiltonian derivations $\{a, -\}$.

As a consequence of [GG02] (see also [Los10, §2.6]), there is a natural action of the stabilizer $\mathrm{Stab}_G(e, h, f)$ of the \mathfrak{sl}_2 -triple (e, h, f) on \mathcal{W}_e by Poisson automorphisms. This is because of the alternative construction of \mathcal{W}_e in [GG02] which is invariant under $\mathrm{Stab}_G(e, h, f)$: $\mathcal{W}_e = (\mathrm{Sym} \mathfrak{g} / \mathfrak{n}'_e \cdot \mathrm{Sym} \mathfrak{g})^{\mathfrak{p}_e}$, where $\mathfrak{n}'_e = \{x - \langle e, x \rangle : x \in \bigoplus_{i \leq -2} \mathfrak{g}_i\}$ and $\mathfrak{p}_e = \bigoplus_{i \leq -1} \mathfrak{g}_i$.

Since this action on \mathcal{W}_e is Hamiltonian, it gives rise to an action of $\pi_0(\mathrm{Stab}_G(e, h, f))$ on $\mathrm{HP}_0(\mathcal{W}_e)$. Note that, since $\mathrm{Stab}_G(e, h, f)$ is the reductive part of $\mathrm{Stab}_G(e)$, the component group coincides with $\pi_0(\mathrm{Stab}_G(e))$. Clearly, this group also acts on $\mathrm{HP}_0(\mathcal{W}_e^0)$.

Our first main result is the following theorem.

Theorem 1.6. As $\pi_0(\text{Stab}_G(e))$ -representations,

$$(1.7) \quad \text{HP}_0(\mathcal{W}_e^0) \cong H^{\dim_{\mathbb{R}}(\rho^{-1}(e))}(\rho^{-1}(e)) \cong \bigoplus_{\chi \in \text{Irrep}_e(W)} \psi_{\chi} \otimes V_{\chi}.$$

Here the action on the right hand side is in the first component.

Remark 1.8. There is a slightly different way to view the Springer correspondence through [HK84] which further explains the above results. Namely, for a smooth variety X , denote by Ω_X the right D -module of volume forms on X . Then [HK84, Theorem 5.3] states that

$$(1.9) \quad \rho_*(\Omega_{\tilde{N}}) \cong \bigoplus_{\chi \in \text{Irrep}(W)} \mathcal{M}_{\chi} \otimes V_{\chi},$$

where \mathcal{M}_{χ} are irreducible, holonomic, pairwise nonisomorphic G -equivariant right D -modules on \mathcal{N} .

Each D -module \mathcal{M}_{χ} is uniquely determined by its support, which is the closure, $\overline{\mathbb{O}}$, of a nilpotent coadjoint orbit $\mathbb{O} = \mathbb{O}(e) \subset \mathcal{N}$ (i.e., a symplectic leaf of \mathcal{N}), together with a G -equivariant local system on \mathbb{O} (the restriction of \mathcal{M}_{χ} to \mathbb{O}). Then, $\mathbb{O} = \mathbb{O}_{\chi}$, and the local system is ψ_{χ} .

Taking the pushforward of (1.9) to a point, one can deduce that $\text{HP}_0(\mathcal{W}_e^0)$ is isomorphic to the RHS of (1.7) using the method of [ES09] recalled in §2 below.

Next, let $U\mathfrak{g}$ denote the universal enveloping algebra of \mathfrak{g} , and let $\mathcal{W}_e^q := (U\mathfrak{g}/\mathfrak{m}'_e U\mathfrak{g})^{\mathfrak{m}_e}$ be the quantum \mathcal{W} -algebra, which is a filtered (in general, noncommutative) algebra whose associated graded algebra is \mathcal{W}_e , as a Poisson algebra. The center of \mathcal{W}_e^q is an isomorphic image of $Z(U\mathfrak{g})$, which is identified with Z as an algebra via the Harish-Chandra isomorphism. Let $\eta : Z \rightarrow \mathbb{C}$ be a character, and define the algebras $\mathcal{W}_e^{\eta} := \mathcal{W}_e^q/(\ker(\eta))$ and $\mathcal{W}_e^{q,\eta} := \mathcal{W}_e^q/(\ker(\eta))$. These are filtered Poisson (respectively, associative) algebras whose associated graded algebras are \mathcal{W}_e^0 . Moreover, using the construction of [GG02] as above (i.e., $\mathcal{W}_e^q \cong (U\mathfrak{g}/\mathfrak{n}'_e U\mathfrak{g})^{\mathfrak{p}_e}$), \mathcal{W}_e^q as well as $\mathcal{W}_e^{q,\eta}$ admit actions of $\text{Stab}_G(e, h, f)$ (as does \mathcal{W}_e^{η} , for all η). Since this action is Hamiltonian, $\text{HP}_0(\mathcal{W}_e^{\eta})$ and $\text{HH}_0(\mathcal{W}_e^{q,\eta})$ admit actions of $\pi_0(\text{Stab}_G(e)) = \pi_0(\text{Stab}_G(e, h, f))$ for all η .

Consider the zeroth Hochschild homology $\text{HH}_0(\mathcal{W}_e^{q,\eta}) := \mathcal{W}_e^{q,\eta}/[\mathcal{W}_e^{q,\eta}, \mathcal{W}_e^{q,\eta}]$. There is a canonical surjection $\text{HP}_0(\mathcal{W}_e^0) \twoheadrightarrow \text{gr HH}_0(\mathcal{W}_e^{q,\eta})$.

Theorem 1.10. (i) The canonical surjection $\text{HP}_0(\mathcal{W}_e^0) \xrightarrow{\sim} \text{gr HH}_0(\mathcal{W}_e^{q,\eta})$ is an isomorphism.

(ii) The families $\text{HP}_0(\mathcal{W}_e^{\eta})$ and $\text{HH}_0(\mathcal{W}_e^{q,\eta})$ are flat in η . In particular, for all η , they are isomorphic to the top cohomology of the Springer fiber, $H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$, as representations of $\pi_0(\text{Stab}_G(e))$.

(iii) The groups $\text{HP}_0(\mathcal{W}_e)$ and $\text{HH}_0(\mathcal{W}_e^q)$ are isomorphic to $Z \otimes H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$ as $Z[\pi_0(\text{Stab}_G(e))]$ -modules.

Theorem 1.10 follows from Theorem 1.13, as explained below.

Corollary 1.11. (C. Dodd, [Dod10]) For every central character η , the number of distinct irreducible finite-dimensional representations of $\mathcal{W}_e^{q,\eta}$ is at most $\dim H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$.

Proof. This immediately follows from the above theorem, because the number of isomorphism classes of irreducible finite-dimensional representations of any associative algebra A is dominated by $\dim A/[A, A]$ (since characters of nonisomorphic irreducible representations are linearly independent functionals on $A/[A, A]$). \square

Remark 1.12. The argument in the appendix to [ES09] by I. Losev together with Corollary 1.11 also implies an upper bound on the number N_e of prime (or, equivalently, primitive) ideals in $\mathcal{W}_e^{q,\eta}$. For every nilpotent orbit $\mathbb{O}_{e'}$ whose closure contains e , let $M_{e,e'}$ denote the number of irreducible components of the intersection $\overline{\mathbb{O}_{e'}} \cap \text{Spec } \mathcal{W}_e$ of the closure of the orbit $\mathbb{O}_{e'}$ with the Kostant-Slodowy slice to e . Then,

$$N_e \leq \sum_{\overline{\mathbb{O}_{e'}} \ni e} M_{e,e'} \cdot \dim H^{\dim_{\mathbb{R}} \rho^{-1}(e')}(\rho^{-1}(e')),$$

where the sum is over the distinct orbits $\mathbb{O}_{e'}$ whose closure contains e . Briefly, we explain this as follows: Losev's appendix to [ES09] gives a map from finite-dimensional irreducible representations of $\mathcal{W}_{e'}^{q,\eta}$ to prime ideals of $\mathcal{W}_e^{q,\eta}$ supported on the irreducible component of $\mathbb{O}_{e'} \cap \text{Spec } \mathcal{W}_e$ containing e' , and shows that all prime ideals are constructed in this way. (More precisely, in *op. cit.*, a construction is given of all prime ideals of filtered quantizations of affine Poisson varieties with finitely many symplectic leaves, which specializes to this one since the aforementioned irreducible components coincide with the symplectic leaves of $\text{Spec } \mathcal{W}_e^0$, and $\mathcal{W}_e^{q,\eta}$ and $\mathcal{W}_{e'}^{q,\eta}$ are quantizations of \mathcal{W}_e^0 and $\mathcal{W}_{e'}^0$, respectively.) Then, the bound follows from Corollary 1.11.

1.4. Higher homology. Finally, following [ES09], one may consider the higher *Poisson-de Rham homology* groups, $\text{HP}_i^{DR}(\mathcal{W}_e^\eta)$, of \mathcal{W}_e^η , whose definition we recall in the following section. Here, we only need to know that $\text{HP}_0^{DR}(A) = \text{HP}_0(A)$ for all Poisson algebras A (although the same is *not* true for higher homology groups). Let $\eta : Z \rightarrow \mathbb{C}$ be an *arbitrary* central character.

The following theorem is a direct generalization of Theorem 1.6. Hence, we only prove this theorem, and omit the proof of the aforementioned theorem.

Theorem 1.13. As $\pi_0(\text{Stab}_G(e))$ -representations, $\text{HP}_i^{DR}(\mathcal{W}_e^\eta) \cong H^{\dim_{\mathbb{R}} \rho^{-1}(e)-i}(\rho^{-1}(e))$. Moreover, for generic η , $\text{HH}_i(\mathcal{W}_e^{q,\eta})$ is also isomorphic to these.

Remark 1.14. For $e = 0$ one has $\mathcal{W}_e^q = U\mathfrak{g}$, and the algebras $\mathcal{W}_e^{q,\eta}$ are the maximal primitive quotients of $U\mathfrak{g}$. In this case, the last statement of Theorem 1.13 holds for all regular characters η (see [Soe96] and [VdB98, VdB02]). On the other hand, the genericity assumption for η cannot be removed for $i > 0$. Namely, for non-regular values of η , it is, in general, *not* true that $\text{HH}_i(\mathcal{W}_e^{q,\eta})$ is isomorphic to the cohomology $H^{\dim_{\mathbb{R}} \rho^{-1}(e)-i}(\rho^{-1}(e))$ of the Springer fiber. For example, when $e = 0$ in $\mathfrak{g} = \mathfrak{sl}_2$, then the variety $\text{Spec } \mathcal{W}_e^0$ is the cone $\mathbb{C}^2/\mathbb{Z}_2$. In this case, by [FSSÁ03, Theorem 2.1] (and the preceding comments), $\text{HH}_i(\mathcal{W}_e^{q,\eta}) \neq 0$ for all $i \geq 3$ when $\eta : Z(U\mathfrak{g}) \rightarrow \mathbb{C}$ is the special central character corresponding to the Verma module with highest weight -1 , i.e., the character corresponding to the fixed point of the Cartan \mathfrak{h} under the shifted Weyl group action.

2. THE CONSTRUCTION OF [ES09]

We prove Theorem 1.13 using the method of [ES09], which we now recall.

To a smooth affine Poisson variety X , we attached the right D -module M_X on X defined as the quotient of the algebra of differential operators D_X by the right ideal generated by Hamiltonian vector fields. Then, $\text{HP}_0(\mathcal{O}_X)$ identifies with the (underived) pushforward $M_X \otimes_{D_X} \mathcal{O}_X$ of M_X to a point.

More generally, if X is not necessarily smooth, but equipped with a closed embedding $i : X \hookrightarrow V$ into a smooth affine variety V (which need not be Poisson), we defined the right D -module $M_{X,i}$ on V as the quotient of D_V by the right ideal generated by functions on V vanishing on X and vector fields on V tangent to X which restrict on X to Hamiltonian vector fields. This is independent of the choice of embedding, in the sense that the resulting D -modules on V supported on X correspond to the same D -module on X (up to a canonical isomorphism) via Kashiwara's theorem. Call this D -module M_X . The pushforward of M_X to a point remains isomorphic to $\mathrm{HP}_0(\mathcal{O}_X)$.

More generally, for an arbitrary affine variety X , we defined the groups $\mathrm{HP}_i^{DR}(\mathcal{O}_X)$ as the full (left derived) pushforward of M_X to a point.

3. PROOF OF THEOREM 1.13

Our main tool is

Theorem 3.1. [HK84, Theorem 4.2 and Proposition 4.8.1.(2)] (see also [LS97, §7])

$$(3.2) \quad M_{\mathcal{N}} \cong \rho_*(\Omega_{\tilde{\mathcal{N}}}).$$

We now begin the proof of Theorem 1.13. First, take $\eta = 0$. Since $M_{\mathcal{N}} \cong \rho_*(\Omega_{\tilde{\mathcal{N}}})$, it follows that, letting π and $\tilde{\pi}$ denote the projections of \mathcal{N} and $\tilde{\mathcal{N}}$ to a point,

$$\mathrm{HP}_i^{DR}(\mathcal{O}_{\mathcal{N}}) = L_i\pi_*(M_{\mathcal{N}}) \cong L_i\tilde{\pi}_*(\Omega_{\tilde{\mathcal{N}}}) \cong H^{\dim_{\mathbb{C}} \tilde{\mathcal{N}}-i}(\tilde{\mathcal{N}}).$$

Similarly, if we consider $\mathrm{Spec} \mathcal{W}_e^0 \subseteq \mathcal{N}$ (the intersection of a Kostant-Slodowy slice to the orbit of e with \mathcal{N}), then it follows, viewing all varieties as embedded in the smooth variety \mathfrak{g} , that $\rho_*(\Omega_{\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)}) \cong M_{\mathrm{Spec} \mathcal{W}_e^0}$, since $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0) \subset \tilde{\mathcal{N}}$ is smooth. In more detail, let Y_e denote a formal neighborhood of $\mathrm{Spec} \mathcal{W}_e^0$ in \mathcal{N} , \tilde{Y}_e denote a formal neighborhood of $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)$ in $\tilde{\mathcal{N}}$, and $\widehat{[e, \mathfrak{g}]}$ denote a formal completion of $[e, \mathfrak{g}]$ at 0. Then, $\tilde{Y}_e \cong \rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0) \times \widehat{[e, \mathfrak{g}]}$ and $Y_e \cong \mathrm{Spec} \mathcal{W}_e^0 \times \widehat{[e, \mathfrak{g}]}$. With these identifications, $\rho|_{\tilde{Y}_e} = \rho|_{\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)} \times \mathrm{Id}_{\widehat{[e, \mathfrak{g}]}}$. Then, $\Omega_{\tilde{Y}_e} \cong \Omega_{\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)} \boxtimes \Omega_{\widehat{[e, \mathfrak{g}]}}$ and $M_{Y_e} \cong \Omega_{\mathrm{Spec} \mathcal{W}_e^0} \boxtimes \Omega_{\widehat{[e, \mathfrak{g}]}}$. Since $M_{\mathcal{N}} \cong \rho_*(\Omega_{\tilde{\mathcal{N}}})$, restricting to \tilde{Y}_e yields $M_{Y_e} \cong \rho_*(\Omega_{\tilde{Y}_e})$, and we conclude from the above that $M_{\mathrm{Spec} \mathcal{W}_e^0} \cong \rho_*(\Omega_{\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)})$.

Since $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0) \rightarrow \mathrm{Spec} \mathcal{W}_e^0$ is birational, $\dim_{\mathbb{C}} \rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0) = \dim_{\mathbb{C}} \mathrm{Spec} \mathcal{W}_e^0$. Hence, $\mathrm{HP}_i^{DR}(\mathcal{W}_e^0) \cong H^{\dim_{\mathbb{C}} \mathrm{Spec} \mathcal{W}_e^0-i}(\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0))$. Next, observe that the contracting \mathbb{C}^* -action on $\mathrm{Spec} \mathcal{W}_e^0$ lifts to a deformation retraction of $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0)$ to $\rho^{-1}(e)$, as topological spaces (in the complex topology). Moreover, $\rho^{-1}(e)$ is compact, and hence its dimension must equal the degree of the top cohomology, $\dim_{\mathbb{R}} \rho^{-1}(e) = \dim_{\mathbb{C}} \rho^{-1}(\mathrm{Spec} \mathcal{W}_e^0) = \dim_{\mathbb{C}} \mathrm{Spec} \mathcal{W}_e^0$. (This can also be computed directly: all of these quantities are equal to the complex codimension of $G \cdot e$ inside \mathcal{N} .) We conclude the first equality of the theorem for $\eta = 0$, i.e., $\mathrm{HP}_i^{DR}(\mathcal{W}_e^0) \cong H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$.

Since the parameter space of η has a contracting \mathbb{C}^* action with fixed point $\eta = 0$, to prove flatness of $\mathrm{HP}_i^{DR}(\mathcal{W}_e^\eta)$, it suffices to show that $\dim \mathrm{HP}_i^{DR}(\mathcal{W}_e^\eta) = \dim \mathrm{HP}_i^{DR}(\mathcal{W}_e^0)$ for generic η . For generic η , $\mathrm{Spec} \mathcal{W}_e^\eta$ is smooth and symplectic, and hence (by [ES09, Example 2.2]), $M_{\mathrm{Spec} \mathcal{W}_e^\eta} = \Omega_{\mathrm{Spec} \mathcal{W}_e^\eta}$, so that $\mathrm{HP}_i^{DR}(\mathcal{W}_e^\eta) \cong H^{\dim \mathrm{Spec} \mathcal{W}_e^\eta-i}(\mathrm{Spec} \mathcal{W}_e^\eta)$. Moreover, $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^\eta) \xrightarrow{\rho} \mathrm{Spec} \mathcal{W}_e^\eta$. Next, for all η , the family $\rho^{-1}(\mathrm{Spec} \mathcal{W}_e^\eta)$ is topologically trivial [Slo80a] (see also [Slo80b]), and hence its cohomology has constant dimension, and equals

$\dim H^{\dim_{\mathbb{C}} \text{Spec } \mathcal{W}_e^0 - i}(\rho^{-1}(\text{Spec } \mathcal{W}_e^0))$. Hence, for generic η , $\dim \text{HP}_i^{DR}(\mathcal{W}_e^\eta) = \dim \text{HP}_i^{DR}(\mathcal{W}_e^0)$, as desired.

Let us now prove the second statement of the theorem. Let \hbar be a formal parameter. For any central character $\eta : Z \rightarrow \mathbb{C}$, consider the character $\eta/\hbar : Z((\hbar)) \rightarrow \mathbb{C}((\hbar))$. Let $\mathcal{W}_e^{q, \eta/\hbar} := \mathcal{W}_e^q((\hbar))/\ker(\eta/\hbar)$. As we will explain below, by results of Nest-Tsygan and Brylinski, for generic η , $\text{HH}_i(\mathcal{W}_e^{q, \eta/\hbar}) = \text{HP}_i(\mathcal{W}_e^\eta)((\hbar)) = H^{\dim_{\mathbb{C}} \mathcal{W}_e^\eta - i}(\text{Spec } \mathcal{W}_e^\eta, \mathbb{C}((\hbar)))$. Hence, $\text{HH}_i(\mathcal{W}_e^{q, \eta/\hbar}) \cong \text{HP}_i^{DR}(\mathcal{W}_e^0)((\hbar))$ for generic η . This implies the statement.¹

In more detail, $\mathcal{W}_e^{q, \eta/\hbar}$ is obtained from a deformation quantization of \mathcal{W}_e^η in the following way. Let \mathcal{W}_e^\hbar be the \hbar -adically completed Rees algebra $\widehat{\bigoplus_{m \geq 0} \hbar^m F^\bullet \mathcal{W}_e^q}$, where $F^\bullet \mathcal{W}_e^q$ is the filtration on \mathcal{W}_e^q . This is a deformation quantization of \mathcal{W}_e . Consider the quotient $\mathcal{W}_e^{\hbar, \eta} := \mathcal{W}_e^\hbar/(\ker(\eta))$. Then, $\mathcal{W}_e^{\hbar, \eta}$ is a deformation quantization of \mathcal{W}_e^η . (Recall that, in general, a deformation quantization of a Poisson algebra A_0 is an algebra of the form $A_\hbar = (A_0[[\hbar]], \star)$, $A_0[[\hbar]] := \{\sum_{i \geq 0} a_i \hbar^i, a_i \in A_0\}$ satisfying $a \star b = ab + O(\hbar)$ and $a \star b - b \star a = \hbar\{a, b\} + O(\hbar^2)$, up to an isomorphism.) Then, by [Bry88, Theorems 2.2.1 and 3.1.1] and [NT95, Theorem A2.1], if \mathcal{W}_e^η is smooth and symplectic, then $\text{HH}_i(\mathcal{W}_e^{\hbar, \eta}[\hbar^{-1}]) \cong \text{HP}_i(\mathcal{W}_e^\eta)((\hbar)) \cong H^{\dim_{\mathbb{C}} \mathcal{W}_e^\eta - i}(\text{Spec } \mathcal{W}_e^\eta, \mathbb{C}((\hbar)))$. Furthermore, the map defined by $x \mapsto \hbar x$, $x \in \mathfrak{g}$ defines an isomorphism $\mathcal{W}_e^{\hbar, \eta}[\hbar^{-1}] \xrightarrow{\sim} \mathcal{W}_e^{\eta/\hbar}$. Since \mathcal{W}_e^η is smooth for generic η , this implies the results claimed in the previous paragraph.

4. PROOF OF THEOREM 1.10

Note that (iii) easily follows from (ii), since $\text{HP}_0(\mathcal{W}_e)$ is a finitely generated \mathbb{Z}_+ -graded Z -module, which is flat (i.e., projective) by Theorem 1.13.

Thus, it suffices to prove that $\dim \text{HH}_0(\mathcal{W}_e^{q, \eta}) = \dim H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$ for all central characters η . As remarked before the statement of Theorem 1.10, there is a canonical surjection $\text{HP}_0(\mathcal{W}_e^0) \twoheadrightarrow \text{gr HH}_0(\mathcal{W}_e^{q, \eta})$. Hence, for all η , $\dim \text{HH}_0(\mathcal{W}_e^{q, \eta}) \leq \dim \text{HP}_0(\mathcal{W}_e^0)$, which equals $\dim H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$ by Theorem 1.6 (which follows from Theorem 1.13). The minimum value of $\dim \text{HH}_0(\mathcal{W}_e^{q, \eta})$ is attained for generic η (since $\text{HH}_0(\mathcal{W}_e^q)$ is a finitely generated Z -module), where it is also $\dim H^{\dim_{\mathbb{R}} \rho^{-1}(e)}(\rho^{-1}(e))$ by Theorem 1.13. Hence, this dimension must be the same for all η .

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REFERENCES

- [Bry88] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. **28** (1988), no. 1, 93–114.
- [Dod10] C. Dodd, 2010, Work in progress.
- [EG02] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.
- [ES09] P. Etingof and T. Schedler, *Poisson traces and D-modules on Poisson varieties*, arXiv:0908.3868, with an appendix by I. Losev, 2009.

¹A similar argument is used in the proof of [EG02, Theorem 1.8.(ii)].

- [FSSÁ03] M. A. Farinati, A. Solotar, and M. Suárez-Álvarez, *Hochschild homology and cohomology of generalized Weyl algebras*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 2, 465–488.
- [GG02] Wee Liang Gan and Victor Ginzburg, *Quantization of Slodowy slices*, Int. Math. Res. Not. (2002), no. 5, 243–255.
- [HK84] R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Invent. Math. **75** (1984), 327–358.
- [Los10] I. Losev, *Finite W-algebras*, arXiv:1003.5811, 2010.
- [LS97] T. Levasseur and J. T. Stafford, *Semi-simplicity of invariant holonomic systems on a reductive Lie algebra*, Amer. J. Math. **119** (1997), 1095–1117.
- [NT95] R. Nest and B. Tsygan, *Algebraic index theorem*, Comm. Math. Phys. **172** (1995), no. 2, 223–262.
- [Slo80a] Peter Slodowy, *Four lectures on simple groups and singularities*, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, vol. 11, Rijksuniversiteit Utrecht Mathematical Institute, Utrecht, 1980.
- [Slo80b] ———, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980.
- [Soe96] W. Soergel, *The Hochschild cohomology ring of regular maximal primitive quotients of enveloping algebras of semisimple Lie algebras*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 4, 535–538.
- [Spr78] T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), no. 3, 279–293.
- [VdB98] Michel Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348.
- [VdB02] ———, *Erratum to: “A relation between Hochschild homology and cohomology for Gorenstein rings”* [Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348; MR1443171 (99m:16013)], Proc. Amer. Math. Soc. **130** (2002), no. 9, 2809–2810 (electronic).